Reconstruction of the S-Matrix for a 3-Port Using Measurements at Only Two Ports

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***Abstract*—A complete S-parameter characterization of a 3-port device is extracted from vector network analyzer measurements made on only two of its ports. The third port is terminated with three known standards, appropriate to the associated transmission medium. Unique extraction of the 3-port matrix from the measured data sets is shown to be possible. This method is particularly useful for devices having incompatible ports, some of which cannot be directly connected to the analyzer.**

I. Introduction

PRECISE experimental characterization of 3-port devices, such as couplers, power-dividers, T-junctions, etc. requires de-embedding of the actual device from the transitions used to connectorize it. This is particularly difficult when some of the device ports cannot be directly accessed or connected to the network analyzer. Examples include elements containing planar transmission lines, or devices incorporating different transmission media (e.g. waveguide and microstrip). In the procedure to be outlined, only two of the three ports need to be connectorized. Consequently *a single two-port calibration procedure* is needed. Three sets of two-port scattering parameters are measured at the chosen terminals, while the third port is terminated in three well-characterized standards. It is shown that the full 3-port scattering matrix can be uniquely reconstructed from the available data. Measurement techniques for 2-port devices with inaccessible and/or noninsertable ports were described in [1], [2].

II. Derivation

Let ports "1" and "2" of a 3-port be chosen for measurement access. Port "3" is terminated in a series of three known (possibly frequency dependent) loads, represented by reflection coefficients Γ_α, where $\alpha = A, B, C$ is the measurement set index. The corresponding measured two-port parameters are defined as follows:

$$\mathcal{S}_{\alpha ij} = \mathcal{S}_{ij}\big|_{a_i=0,\, a_3=b_3\Gamma_\alpha}\ ;\ i,j = 1,2. \tag{1}$$

The sought 3-port S-parameters will be denoted by S_{ij} , $i,j = 1,3$. Symmetry of the scattering matrices is assumed, i.e., $\mathrm{S}_{ij} = \mathrm{S}_{ji}$, $i,j = 1,3$ and $\mathcal{S}_{ij} = \mathcal{S}_{ji}$, $i,j = 1,2$.

Through algebraic manipulation, the following set of relationships between the measured and the 3-port S-parameters is obtained:

$$\mathcal{S}_{\alpha 11} = \mathrm{S}_{11} + \frac{\mathrm{S}_{13}\mathrm{S}_{13}\Gamma_\alpha}{1-\mathrm{S}_{33}\Gamma_\alpha} \tag{2}$$

$$\mathcal{S}_{\alpha 12} = \mathrm{S}_{12} + \frac{\mathrm{S}_{13}\mathrm{S}_{23}\Gamma_\alpha}{1-\mathrm{S}_{33}\Gamma_\alpha} \tag{3}$$

$$\mathcal{S}_{\alpha 22} = \mathrm{S}_{22} + \frac{\mathrm{S}_{23}\mathrm{S}_{23}\Gamma_\alpha}{1-\mathrm{S}_{33}\Gamma_\alpha}. \tag{4}$$

The objective of the following analysis is to present formulas for the S_{ij} in terms of $\mathcal{S}_{\alpha ij}$; $\alpha = A, B, C$.

Examination of the preceding set of equations reveals that the unknowns, i.e., various unique combinations of the 3-port S-parameters, number seven, while there are nine equations in total. Therefore, it is unclear whether a unique solution is possible. This important issue will be resolved affirmatively in the course of analysis.

The parameter S_{33} appears in all equations and is therefore considered key to solving the problem. In order to derive an expression for S_{33} and prove its uniqueness, the equations are rewritten more concisely, as follows:

$$\mathcal{S}_{\alpha ij} = \mathrm{S}_{ij} + \mathrm{S}_{13}^2\lambda^{i+j-2}\gamma_\alpha \tag{5}$$

where $i,j = 1,2$; $\alpha = A,B,C$ and

$$\lambda \equiv \frac{\mathrm{S}_{23}}{\mathrm{S}_{13}} \tag{6}$$

$$\gamma_\alpha \equiv \frac{\Gamma_\alpha}{1-\mathrm{S}_{33}\Gamma_\alpha}. \tag{7}$$

Let the index set (i,j) be fixed, and consider the differences between the equations generated by (5) with $\alpha = A,B,C$. Three distinct equations are obtained

$$\mathcal{S}_{Aij} - \mathcal{S}_{Bij} = \mathrm{S}_{13}^2\lambda^{i+j-2}(\gamma_A - \gamma_B) \tag{8}$$

$$\mathcal{S}_{Aij} - \mathcal{S}_{Cij} = \mathrm{S}_{13}^2\lambda^{i+j-2}(\gamma_A - \gamma_C) \tag{9}$$

$$\mathcal{S}_{Bij} - \mathcal{S}_{Cij} = \mathrm{S}_{13}^2\lambda^{i+j-2}(\gamma_B - \gamma_C). \tag{10}$$

It is observed that one of these equations is redundant—it is either the sum or the difference of the remaining two. Consequently, unknown parameters extracted from any two of these equations will satisfy the third. Division of the left- and right-hand sides of (8) by the respective parts of (9) yields

$$\frac{\mathcal{S}_{Aij} - \mathcal{S}_{Bij}}{\mathcal{S}_{Aij} - \mathcal{S}_{Cij}} = \frac{\gamma_A - \gamma_B}{\gamma_A - \gamma_C} \tag{11}$$

which, after substitution for γ_α from (7), can be solved for S_{33}. The result is given by

$$\mathrm{S}_{33} = \frac{1-\mathcal{U}}{\Gamma_C - \Gamma_B\mathcal{U}} \tag{12}$$

where

$$\mathcal{U} = \left(\frac{\mathcal{S}_{Aij} - \mathcal{S}_{Bij}}{\mathcal{S}_{Aij} - \mathcal{S}_{Cij}}\right)\left(\frac{\Gamma_A - \Gamma_C}{\Gamma_A - \Gamma_B}\right) \tag{13}$$

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TABLE I
S-PARAMETER COMPARISON

Method	S_{11}	S_{12}	S_{22}	S_{13}	S_{23}	S_{33}
Direct	.1024 $\angle 0.4845°$	.09933 $\angle 82.64°$	.1124 $\angle -2.901°$	.6755 $\angle -56.70°$	.6692 $\angle -53.56°$	.1539 $\angle 27.66°$
Proposed	.1024 $\angle 0.4845°$	.09928 $\angle 82.82°$	.1138 $\angle -2.256°$	.6787 $\angle -56.87°$	.6707 $\angle -53.75°$	.1590 $\angle 29.09°$

It is important to prove that in the *absence of experimental errors*, (12) represents a *unique* solution, independent of the choice of measurement set α or port-index (i, j) combinations. Although a formal proof is available, its detailed exposition is deemed unessential for practical implementation. Briefly, the issue of uniqueness with respect to the dataset α selection was touched upon in the sequel to equations (8)–(10). Moreover, further examination of these equations reveals the combination of the measured S-parameters appearing in (13), and in turn in (12), to be independent of (i, j).

Equations (8)–(10) can also be used to compute S_{13} and S_{23}, e.g. from (8)

$$S_{13}^2 = \frac{\mathcal{S}_{A11} - \mathcal{S}_{B11}}{\gamma_A - \gamma_B} \tag{14}$$

$$\lambda = \frac{S_{23}}{S_{13}} = \frac{\mathcal{S}_{A12} - \mathcal{S}_{B12}}{\mathcal{S}_{A11} - \mathcal{S}_{B11}} \tag{15}$$

where S_{33} is now assumed known. It should be noted that S_{13} and S_{23} are determined up to a common 180° phase. However, the phase difference between these two parameters, which is typically the more important quantity, is invariant. Such indeterminacy is common in calibration procedures and can be resolved by a crude electrical length estimate [1].

The remaining parameters can be found directly from (2)–(4) with an arbitrary value of α, e.g.

$$S_{11} = \mathcal{S}_{A11} - \frac{S_{13}S_{13}\Gamma_A}{1 - S_{33}\Gamma_A} \tag{16}$$

$$S_{12} = \mathcal{S}_{A12} - \frac{S_{13}S_{23}\Gamma_A}{1 - S_{33}\Gamma_A} \tag{17}$$

$$S_{22} = \mathcal{S}_{A22} - \frac{S_{23}S_{23}\Gamma_A}{1 - S_{33}\Gamma_A}. \tag{18}$$

III. VERIFICATION AND CONCLUSION

To validate the proposed technique, measurements were carried out on a 8–12GHz packaged 3-dB power divider, with SMA input-output ports. The Hewlett-Packard 8510C network analyzer with the 3.5-mm calibration set was used.

Results of two measurement approaches were compared. In the "direct" method, two ports were accessed at a time, while the third was match-terminated. The other set of results was obtained by making measurements on the two "output" arms of the divider, while terminating the third ("input") with a load, short, and an open taken from the 3.5-mm calibration set. The characteristics of the terminations were specified by Hewlett-Packard. Comparison of the 3-port scattering matrices, obtained at 9.0 GHz by the two aforementioned methods, is summarized in Table I. The agreement between the values is well within the limits imposed by the uncertainties in the termination characteristics and calibration procedures.

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